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Existence of Regular Solution and Attractor for Tearing Mode Instabilities with a Resistivity Depending on a Flux Function

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Abstract—We consider tearing mode instabilities when the resistivity depends on a flux function (ψ) for a bidimensional layer of plasma. This problem modelized by M.H.D. equations is written in terms of flux functions, and in this work, we first show, using a fixed-point method, existence of a local regular solution of the considered problem. Next we show existence of a global solution, and we end this paper with existence of a global attractor. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

We are concerned in this work with time-depending tearing-mode instabilities in slab geometry, when the resistivity depends on a flux function. In a preceding paper (see [1], and references therein), we studied bifurcations of steady solutions for that problem.

The study of tearing instabilities modelized by M.H.D. equations is of great importance due to the difficulty to avoid them in fusion experiments. Many results, either theoretical or numerical, have been carried out on these instabilities.

In general, hot plasma's resistivity depends on the temperature (T_e), but in most of the mentioned works [2–5], the resistivity is considered either as constant or as depending only on a space variable.

Due to the high diffusion coefficient of this temperature in the parallel direction to the magnetic field, T_e can be considered as constant per magnetic surface (which is modelized by the equation ψ = constant, where ψ is a flux function). In relation with this observation, we consider the resistivity as depending nonlinearly on a flux function.

Our aim in this paper is to show existence of a regular solution and existence of a global attractor for the full nonlinear tearing mode instabilities in slab geometry when the resistivity depends on ψ. For so doing, after a brief description of the equations and of the functional spaces, we first show, using a fixed-point method, the existence of a local solution. Next, we obtain estimates for existence of a global solution, and at last we show existence of a global attractor.

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Following notations $|u|$, $\|u\|$, $\|u\|_p$, $\|u\|_{k,p}$, $|||u\,|||_{s,k,p}$ design norm of u in $L^2(\Omega), H^1(\Omega), L^p(\Omega)$, $W^{k,p}(\Omega)$, and in $L^s(0,T;W^{k,p}(\Omega))$, respectively, the same notation is used if u is a vector of $\mathbb{R}^2$ defined on Ω.

2. EQUATIONS

The magnetic field and the velocity can be expressed, in slab geometry, in terms of flux functions $V = e_z \wedge \nabla\phi$ and $B = e_z \wedge \nabla\psi$. We deduce, from the M.H.D. equations, the following equations and boundary conditions for ϕ and ψ, written for a perturbation of a given static equilibrium (ψ_{eq}), where S is the Lundquist number, $\wp_R$ is a Prandtl number, and η is the resistivity (depending on the unknown ψ):

$$\begin{aligned}
-\frac{\partial}{\partial t}\Delta\phi + \wp_R\Delta^2\phi &= S\left(\nabla\phi\wedge\nabla\Delta\phi - \nabla\psi\wedge\nabla\Delta\left(\psi+\psi_{\text{eq}}\right) - \nabla\psi_{\text{eq}}\wedge\nabla\Delta\psi\right), \\
\frac{\partial}{\partial t}\psi - \eta\left(\psi+\psi_{\text{eq}}\right)\Delta\psi &= -S\nabla\phi\wedge\nabla\left(\psi+\psi_{\text{eq}}\right) + \left(\eta\left(\psi+\psi_{\text{eq}}\right) - \eta\left(\psi_{\text{eq}}\right)\right)\Delta\psi_{\text{eq}},
\end{aligned} \tag{1}$$

$$\chi = 0 = \Delta\chi \text{ at } x = \pm\frac{1}{2}; \quad \frac{\partial^j}{\partial y^j}\chi \text{ periodic in } y, \qquad \text{for } j = 0,1,2,3, \qquad \text{with } \chi \in \{\phi,\psi\}. \tag{2}$$

3. FUNCTIONAL SPACES

Set $\Omega =]-1/2, 1/2[\times]0,L[$ and $Q =]0,T[\times\Omega$, $T<\infty$. We define

$$\begin{aligned}
E &= H^2_{\text{per}}(\Omega)\cap H^1_{0,\text{per}}(\Omega), \\
E_1 &= \left\{u\in H^3_{\text{per}}(\Omega)\cap H^1_{0,\text{ per}}(\Omega);\ \Delta u = 0 \text{ at } x = \pm\frac{1}{2}\right\}, \\
E_2 &= \left\{u\in H^4_{\text{per}}(\Omega)\cap H^1_{0,\text{ per}}(\Omega);\ \Delta u = 0 \text{ at } x = \pm\frac{1}{2}\right\}, \\
E_3 &= \left\{u\in H^5_{\text{per}}(\Omega);\Delta^j u = 0 \text{ at } x = \pm\frac{1}{2},\ 0\le j\le 2\right\},
\end{aligned}$$

where (cf. [6,7])

$$\begin{aligned}
H^m_{\text{per}}(\Omega) &= \left\{u\in H^m(\Omega);\ \frac{\partial^j}{\partial y^j}u(x,0) = \frac{\partial^j}{\partial y^j}u(x,L) \text{ for } j = 0, m-1\right\}, \\
H^m_{0,\text{ per}}(\Omega) &= \left\{u\in H^m_{\text{per}}(\Omega);\ \frac{\partial^j}{\partial x^j}u\left(\pm\frac{1}{2},y\right) = 0 \text{ for } j = 0,\ m-1\right\}.
\end{aligned}$$

The norms on $H^1_{0,\text{ per}}(\Omega)$, E, E_1, E_2, E_3, are, respectively, $|\nabla\phi|$, $|\Delta\phi|$, $|\nabla\Delta\phi|$, $|\Delta^2\phi|$, $|\nabla\Delta^2\phi|$. We define

$$\begin{aligned}
C^\infty_{\text{per}}(\overline{\Omega}) &= \left\{u\in C^\infty(\overline{\Omega}) \text{ with } \frac{\partial^j u}{\partial y^j} \text{ periodic for all } j\right\}, \\
\mathcal{E} &= \left\{u\in C^\infty_{\text{per}}(\overline{\Omega}); \Delta^m u = 0 \text{ at } x = \pm\frac{1}{2},\ m\ge 0\right\}.
\end{aligned}$$

The so defined $\mathcal{E}$ is dense in E, E_1, E_2, and E_3. Assume

$$\begin{aligned}
&\eta\in W^{3,\infty}(\mathbb{R}) \text{ and there exists two reals } n_1, n_2, \text{ such that:} \\
&0 < n_1 \le \eta(x) \le n_2 \text{ for all } x\in\mathbb{R};\ \sup|\eta'| \le n_2,\ \sup|\eta''| \le n_2, \text{ and } \sup|\eta'''| \le n_2; \\
&\psi_{\text{eq}}\left(\pm\frac{1}{2}\right) = 0,\ \psi_{\text{eq}}\in C^\infty(\bar{\Omega}) \text{ depending only on } x.
\end{aligned} \tag{3}$$

4. EXISTENCE OF A REGULAR SOLUTION

We first consider the following linear problem (see [8] for the proof).

PROPOSITION 4.1. *Assume* $z = (z_1, z_2) \in (\mathcal{C}([0,T]; E_1))^2$ *and* $u_0 = (\phi_0, \psi_0) \in E_1^2$, *then there exists one and only one* $u = (\phi, \psi)$, $u \in (L^2(0,T;E_2) \cap H^1(0,T;E))^2$ *that verifies equations*

$$-\frac{\partial}{\partial t}\Delta\phi + \wp_R \Delta^2\phi = f_1 \in L^2(Q), \frac{\partial}{\partial t}\psi - \eta(z_2 + \psi_{\text{eq}})\Delta\psi = f_2 \in L^2(0,T;E)$$
$$\phi(0) = \phi_0, \quad \psi(0) = \psi_0, \qquad \textit{in } E_1. \tag{4}$$

PROOF. A standard Galerkin approximation, with a special basis of eigenfunctions, allows us to obtain estimates for a finite-dimensional approximation, and then existence of u, with estimates depending on z_2. ∎

THEOREM 4.2. *Assume (3) and* $u_0 \in E_1^2$, *then for all* $T < \infty$, *there exists one and only one* $u = (\phi, \psi)$,

$$u \in (L^2(E_2) \cap H^1(E))^2 \subset (\mathcal{C}([0,T]; E_1))^2$$

solution of equations (1), with $u(0) = u_0$.

PROOF.

(i) Definition of a Map h for Existence of a Local Solution on $(0, T^*)$

Consider the following decoupled equations, with unknown $u = (\phi, \psi)$, where $\bar{u} = (\bar{\phi}, \bar{\psi})$ is given:

$$-\frac{\partial}{\partial t}\Delta\phi + \wp_R\Delta^2\phi = S\left(\nabla\overline{\phi} \wedge \nabla\Delta\overline{\phi} - \nabla\overline{\psi} \wedge \nabla\Delta\left(\overline{\psi} + \psi_{\text{eq}}\right)\right) - S\nabla\psi_{\text{eq}} \wedge \nabla\Delta\overline{\psi},$$
$$\frac{\partial}{\partial t}\psi - \eta(\overline{\psi} + \psi_{\text{eq}})\Delta\psi = -S\nabla\overline{\phi} \wedge \nabla(\overline{\psi} + \psi_{\text{eq}}) + \left(\eta(\overline{\psi} + \psi_{\text{eq}}) - \eta(\psi_{\text{eq}})\right)\Delta\psi_{\text{eq}},$$
$$\phi(0) = \phi_0, \quad \psi(0) = \psi_0 \qquad \text{in } E_1. \tag{5}$$

Set $X = \mathcal{C}([0,T]; E)^2$, B a real, and

$$\theta_T = \Big\{u = (\phi, \psi) \in \left(L^2(0,T;E_2) \cap H^1(0,T;E)\right)^2;$$
$$[\phi]^2 = \|\phi\|^2_{L^2(0,T;E_2)} + \|\phi'\|^2_{L^2(0,T,E)} \le B, [\psi]^2 = \|\psi\|^2_{L^2(0,T;E_2)} + \|\psi'\|^2_{L^2(0,T,E)} \le B\Big\}.$$

We note that, by interpolation, $L^2(0,T,E_2) \cap H^1(0,T,E) \subset \mathcal{C}([0,T], E_1)$.

For $\bar{u} = (\bar{\phi}, \bar{\psi}) \in \theta_T$, the system (5) associated with boundary conditions (2) admits, due to Hölder inequalities, Sobolev imbeddings, and Proposition 4.1, one and only one solution $u \in (L^2(0,T;,E_2) \cap H^1(0,T;E))^2$. Consider the map $h : \theta_T \longrightarrow X; \bar{u} \longmapsto u$. Using the estimates obtained in Proposition 4.1, we see that there exists a real B and a real T^* (depending on the norm of $u_0 = (\phi_0, \psi_0)$ in E_1^2) such that $h(\theta_{T^*}) \subset \theta_{T^*}$.

More precisely, for $B \ge c \text{ Sup}\left(|\nabla\Delta\phi_0|^2, |\nabla\Delta\psi_0|^2\right)$ and $T \le c \text{ Inf}(1, B)/(B^2 + B + 1)$, one has $[\phi]^2 \le B$ and $[\psi]^2 \le B$ (in these relations, c represents constants independent of B, T, and u_0).

(ii) θ_T is a Compact Subset of X

The following imbedding is compact (see [9]): $L^2(0,T;E_2) \cap H^1(0,T;E) \subset C([0,T];E)$. Moreover, θ_T is closed in X, then is compact.

(iii) Continuity of h for X's Topology

The map h defined from θ_{T^*} to X (for $T = T^*$), which associates u to $\bar{u}$, is continuous for X's topology. Let $\tilde{u}_n$ and $\tilde{u}$ in θ_{T^*}, and $u_n = (\phi_n, \psi_n) = h(\tilde{\phi}_n, \tilde{\psi}_n) = h(\tilde{u}_n)$ and $u = (\phi, \psi) = h(\tilde{\phi}, \tilde{\psi}) = h(\tilde{u})$.

We first prove, using estimates, that if $\tilde{u}_n \to \tilde{u}$ in X, with $\tilde{u}_n$ and $\tilde{u}$ in θ_{T^*}, then $u_n \to u$ in $(L^\infty(0,T^*; H^1_{0,\text{ per}}) \cap L^2(0,T^*;E))^2$.

Moreover, as $u_n = h(\tilde{u}_n) \in \theta_{T^*}$ which is compact in X (see (ii)), there exists a subsequence of u_n which converges in X, then also in $(L^2(0,T^*;E))^2$; the limit is then $u = h(\tilde{u})$ and all the sequence converges in X.

(iv) Application of a Fixed-Point Theorem

The set θ_{T^*} is nonempty, convex, and compact in X (see (ii)). The map h, from θ_{T^*} to θ_{T^*}, is continuous for X topology. Existence of a solution $u \in \theta_{T^*}$ for the system (1),(2) is then given by Schauder's fixed-point theorem [10]. ∎

(v) Uniqueness of the Solution

Let $u_1 = (\phi_1, \psi_1)$ and $u_2 = (\phi_2, \psi_2)$ be two solutions, and set $w = u_1 - u_2 = (v, z)$. Making the difference of the equations of u_1 and u_2, multiplying, respectively, by v and $-\Delta z$, applying Young's inequality, Sobolev imbeddings, and taking into account that u_1 and $u_2 \in \theta_{T^*}$, we obtain, with Gronwall's lemma, $w = 0$.

(vi) Existence of a Global Solution

We have to show that the solution obtained on $[0, T^*]$ can be extended to any interval $[0, T]$, $T* \leq T < \infty$. For so doing, it will be sufficient to show that $\|u\|_{C([0,T^*];E_1)}$, where u is the solution obtained in the first part, is bounded independently of T^*. Indeed, once the solution on $[0, T^*]$ is obtained, one considers $u(T^*)$ as initial condition and by the local existence result of part (iv), there exists T_1^* (with $T_1^* - T^*$ bounded below by a function of some upper bound of the norm of $u(T^*)$ in E_1^2—see above—which will be proved independent of T^*) such that there exists a unique solution $u_1 \in L^2(T^*, T_1^*; E_2) \cap H^1(T^*, T_1^*; E)$ for the considered problem with $u_1(T^*) = u(T^*)$. Therefore, we obtain a solution u of equations (1),(2) (in the sense of distributions on $(0, T_1^*)$) belonging to $H^1(0, T_1^*; E) \cap L^2(0, T_1^*; E_2)$. One repeats this process to get a sequence of reals $T_1^*, T_2^*, \ldots, T_n^*$, until $T_n^* \geq T$. Then it remains to prove that the norm of u in $C([0, T^*]; E_1)$ is bounded independently of T^*.

(a) Multiplying the first and the second equation of (1) by ϕ and $-\Delta\psi$, respectively, adding the results, using Hölder's inequality, and applying Gronwall's lemma, one obtains, with $\mu = \operatorname{Inf}(\mathcal{P}_R, n_1)$,

$$|\nabla u|^2 \leq K_1 = |\nabla u_0|^2 \exp \int_0^T \|\Delta\psi_{\text{eq}}\|_\infty \left(\frac{n_2^2}{\mu} \|\Delta\psi_{\text{eq}}\|_\infty + S \right) d\tau,$$

$\int_0^{T^*} |\Delta u|^2 \leq K_2 = 1/\mu |\nabla u_0|^2 + (1/\mu) K_1 \int_0^T \|\Delta\psi_{\text{eq}}\|_\infty ((n_2^2/\mu)\|\Delta\psi_{\text{eq}}\|_\infty + S)\, d\tau$. We can define quantities K_1 and K_2 for all $T < \infty$.

(b) Same operations as in (a), but in which one replaces ϕ and $-\Delta\psi$ by $-\Delta\phi$ and $\Delta^2\psi$, respectively, taking into account regularity of ψ and η, applying Hölder's inequality and Gagliardo-Nirenberg's interpolation inequality (for $p = 3$ and 4), one also deduces

$$\frac{d}{dt}\left(1 + |\Delta u|^2\right) + \mu |\nabla \Delta u|^2 \leq C\left(1 + |\nabla u|^2\right)\left(1 + |\Delta u|^2\right)^2. \tag{6}$$

With Gronwall's lemma and part (a), we obtain

$$|\Delta u|^2 \leq \left(1 + |\Delta u_0|^2\right) \exp\left(C\left(1 + K_1\right)\left(T + K_2\right)\right) = K_3$$

and

$$\int_0^{T^*} |\nabla \Delta u|^2 \leq C\left(1 + K_1\right)\left(T + K_2\right)\left(1 + K_3\right) = K_4.$$

(c) We now consider analogous products as in (a) and (b): $((1)_1, \Delta^2\phi)$ and $(\Delta((1)_2), -\Delta^2\psi)$. Due to Hölder, Young and Gagliardo-Nirenberg's interpolation inequalities, one has

$$\begin{aligned} \frac{d}{dt}|\nabla \Delta u|^2 + \mu \left|\Delta^2 u\right|^2 \leq C \big(&|\nabla u|^2 \|\psi_{\text{eq}}\|_{2,\infty}^2 + |\Delta u|^2 + |\Delta u|^4 \\ &+ \|\psi_{\text{eq}}\|_{4,\infty}^2 + \|\psi_{\text{eq}}\|_{3,\infty}^4 + \|\psi_{\text{eq}}\|_{2,\infty}^6\big) |\nabla \Delta u|^2, \end{aligned} \tag{7}$$

and using (a) and (b), we have

$$|\nabla\Delta u|^2 \le |\nabla\Delta u_0|^2 + C\left(K_3 + K_3^2 + K_1\|\psi_{\rm eq}\|_{2,\infty}^2 + \|\psi_{\rm eq}\|_{4,\infty}^2 + \|\psi_{\rm eq}\|_{3,\infty}^4 + \|\psi_{\rm eq}\|_{2,\infty}^6\right) K_4 = K_5$$

and $\int_0^{T^*} |\Delta^2 u|^2 \le K_5/\mu = K_6$. We have shown that $\|u\|_{C([0,T^*];E_1)}$ is bounded independently of T^*, and the solution can then be extended to the interval $[0,T]$, $T<\infty$. This ends the proof of Theorem 4.2. ∎

5. ABSORBING SETS AND ATTRACTOR

In this section, we study the existence of an attractor. We use some definitions and general results, which can be found, for example, in [11].

We have shown in Section 4, Theorem 4.2, that equations (1) admit a unique solution u and this is sufficient to justify the existence of a semigroup $\mathcal{S}(t)$, $t\ge 0$ defined on E_1^2 (see [11]) which to a given $u_0 \in E_1^2$ associates the solution of Theorem 4.2. We are now going to show that this semigroup is continuous, the existence of absorbing sets, and the uniform compactness of the operators $\mathcal{S}(t)$.

5.1. Continuity of the Semigroup

PROPOSITION 5.1. *Under the hypothesis of Theorem 4.2, the solution u of equations (1) depends continuously on the initial condition u_0.*

PROOF. Let u_1 and u_2 be two solutions of equations (1) with, respectively, u_0^1 and u_0^2 as initial condition.

Set $v=\phi_2-\phi_1$; $z=\psi_2-\psi_1$. Proceeding as for the estimates obtained to prove uniqueness and existence of global solutions, we have $|||\zeta\,|||_{\infty,3,2} \le C\|\zeta_0\|_{3,2}$ where $\zeta=(v,z)$ and C is bounded is the data are bounded, which implies the continuous dependence of u on u_0, and so that of the semigroup $\mathcal{S}(t)$. ∎

5.2. Existence of Absorbing Sets

PROPOSITION 5.2. *Equations (1) associated with boundary conditions (2) possess an absorbing set in E_1^2, i.e., $\exists\, B\subset E_1^2$ such that the orbits starting from any bounded set of E_1^2 enter into B after a finite time (depending on the set).*

PROOF.

(i) Estimates in $H^1_{0,\,\rm per}(\Omega)$

Multiply the two equations (1) by ϕ and $-\Delta(\psi+\psi_{\rm eq})$, respectively, and add the results. Applying Schwartz's inequality gives (with λ a positive constant and with the same notations μ, u, and $u_{\rm eq}$ as in the proof of Proposition 4.2, (vi)(a)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(|\nabla\phi|^2 + |\nabla(\psi+\psi_{\rm eq})|^2\right) + \mathcal{P}_R|\Delta\phi|^2 + \int_\Omega \eta(\psi+\psi_{\rm eq})\,|\Delta(\psi+\psi_{\rm eq})|^2 \\ = \int_\Omega \eta(\psi_{\rm eq})\,\Delta\psi_{\rm eq}\Delta(\psi+\psi_{\rm eq}),\end{aligned} \tag{8}$$

$\frac{d}{dt}|\nabla(u+u_{\rm eq})|^2 + \lambda\mu|\nabla(u+u_{\rm eq})|^2 \le Cn_2^2|\Delta\psi_{\rm eq}|^2$, and then $|\nabla(u+u_{\rm eq})|^2 \le |\nabla(u_0+u_{\rm eq})|^2 \exp(-\lambda\mu t) + (Cn_2^2/\lambda\mu)|\Delta\psi_{\rm eq}|^2(1-\exp(-\lambda\mu t))$. Thus, $\lim_{t\longrightarrow\infty}\sup|\nabla u(t)| \le \rho_0$, where $\rho_0 = (Cn_2^2/\lambda\mu)^{1/2}|\Delta\psi_{\rm eq}| + |\nabla u_{\rm eq}|$.

So any ball of $(H^1_{0,\,\rm per}(\Omega))^2$ of radius $\rho'_0>\rho_0$ centered at the origin, denoted $\mathcal{B}_0$, has the following property. If $\mathcal{B}$ is a subset of E_1^2 which is bounded in $(H^1_{0,\,\rm per}(\Omega))^2$, included in a ball $B(0,R)$ of $(H^1_{0,\,\mathrm{per}}(\Omega))^2$, then $\mathcal{S}(t)\mathcal{B} \subset \mathcal{B}_0 = B(0,\rho_0')$ in $(H^1_{0,\,\mathrm{per}}(\Omega))^2$, for $t \geq t_0(\mathcal{B}, \rho'_0)$, $t_0 = (2/\lambda\mu)\ln(R + |\nabla u_{\mathrm{eq}}|)/((\rho_0') - \rho_0)$. Moreover, (8) implies $\frac{d}{dt}|\nabla(u + u_{\mathrm{eq}})|^2 + \mu|\Delta u|^2 \leq C_1$, where $C_1 = (10n_2^2/\mu)|\Delta u_{\mathrm{eq}}|^2 + 4n_2|\Delta u_{\mathrm{eq}}|^2$, from which one deduces, by integration between t and $t+r$, where r is a fixed positive number, that, if $u_0 \in \mathcal{B} \subset B(0, R)$ of $(H^1_{0,\,\mathrm{per}}(\Omega))^2$, and $t \geq t_0(\mathcal{B}, \rho'_0)$,

$$\int_t^{t+r} |\Delta u|^2 \leq \frac{1}{\mu}\left(C_1 r + (\rho'_0 + |\nabla u_{\mathrm{eq}}|)^2\right). \tag{9}$$

(ii) Estimates in E

Let $\mathcal{B}$ be a subset of E_1^2 which is bounded in E^2. Then $\mathcal{B}$ is also a bounded set of $(H^1_{0,\,\mathrm{per}}(\Omega))^2$, and (cf. (i)), for $t \geq t_0(\mathcal{B}, \rho'_0)$, we have $|\nabla u(t)| \leq \rho'_0$. Estimates (6) obtained in Section 4 imply, for $t \geq t_0$,

$$\frac{d}{dt}\left(1 + |\Delta u|^2\right) + \mu|\nabla\Delta u|^2 \leq C\left(1 + \rho'^2_0\right)\left(1 + |\Delta u|^2\right)\left(1 + |\Delta u|^2\right).$$

With (9) and the uniform Gronwall's lemma (cf. [11]), we obtain, for $t \geq t_0 + r$,

$$|\Delta u(t)|^2 \leq \frac{a_3}{r}\exp(a_1), \qquad \text{where } a_1 = C\left(1 + \rho'^2_0\right)a_3; \quad a_3 = \frac{1}{\mu}\left(C_1 r + (\rho'_0 + |\nabla u_{\mathrm{eq}}|)^2\right) + r.$$

Set $\rho_1^2 = a_3/r\exp(a_1)$, then the ball of E^2 centered at the origin and of radius ρ_1, denoted $\mathcal{B}_1$, is such that, if $\mathcal{B}$ is bounded in E^2, then $\mathcal{S}(t)\mathcal{B} \subset \mathcal{B}_1$ for all $t \geq t_0(\mathcal{B}, \rho'_0) + r = t_1(\mathcal{B})$.

Integrating (6) on $[t, t+r]$ yields

$$\int_t^{t+r} |\nabla\Delta u|^2 \leq b_3 = \frac{1}{\mu}\left(\rho_1^2 + a_1\left(1 + \rho_1^2\right)\right), \qquad \forall t \geq t_1. \tag{10}$$

(iii) Existence of Absorbing Sets in E_1

Let $\mathcal{B}$ be a bounded set in E_1^2. It is also a bounded set in $(H^1_{0,\,\mathrm{per}}(\Omega))^2$ and E^2; for $t \geq t_1$ defined as in (i) and (ii), we have (9) and (10). Using Gronwall's uniform lemma (cf. [11]), one deduces from estimates (7) and (10) that

$$|\nabla\Delta u(t)|^2 \leq \left(\frac{b_3}{r}\right)\exp(b_1) = \rho_2^2, \qquad \text{for } t \geq t_1 + r, \tag{11}$$

where $b_1 = r\,C(\rho'^2_0\|\psi_{\mathrm{eq}}\|^2_{2,\infty} + \rho_1^2 + \rho_1^4 + \|\psi_{\mathrm{eq}}\|^2_{4,\infty} + \|\psi_{\mathrm{eq}}\|^4_{3,\infty} + \|\psi_{\mathrm{eq}}\|^6_{2,\infty})$. One deduces from (11), the existence of an absorbing set $\mathcal{B}_2 = B(0,\rho_2)$ in E_1^2 for the semigroup $\mathcal{S}(t)$. That is, if $\mathcal{B}$ is a bounded set of E_1^2, then $\mathcal{S}(t)\mathcal{B} \subset \mathcal{B}_2$ for all $t \geq t_1 + r = t_2(\mathcal{B})$.

Integrating (7) on $[t, t+r]$ yields

$$\int_t^{t+r} \left|\Delta^2 u\right|^2 \leq \frac{1}{\mu}\left(\rho_2^2 + b_1\rho_2^2\right) = c_3, \qquad \forall t \geq t_2. \tag{12}$$ ∎

5.3. Uniform Compacity of the Operators $\mathcal{S}(t)$

Let $\mathcal{B}$ be a bounded set in $(E_1)^2$. First, we want to prove some bound of $S(t)\mathcal{B}$ in E_2^2 for large t, for almost all t. Let $u(t) = (\phi(t), \psi(t)) \in \mathcal{S}(t)\mathcal{B}$. Let f_3 and f_4 be the right-hand sides of equations (1) for ϕ and ψ.

For $t \geq t_2(\mathcal{B})$, we have (11) and (12). Then, using Hölder inequalities, and (11), (12), one obtains the following.

LEMMA 5.3. *If $\eta \in W^{4,\infty}(\mathbb{R})$ and if $\eta'(0) = 0$, $\frac{d^2\psi_{\mathrm{eq}}}{dx^2}(\pm 1/2) = 0$, there exists a constant $C(\mathcal{B})$ such that $\forall t \geq t_2(\mathcal{B})$ $\int_t^{t+r} |\nabla\Delta f_4|^2 \leq C$, $\Delta f_4 = 0$ at $x = \pm 1/2$; $\int_t^{t+r} |\nabla f_3|^2 \leq C$, $f_3 = 0$ at $x = \pm 1/2$.*

Due to lack of regularity of the initial condition u_0 ($u_0 \notin E_2^2$), bounds on $\Delta^2\phi(t)$ and $\Delta^2\psi(t)$ will be obtained first on the following Galerkin approximates ϕ_m and ψ_m of ϕ and ψ:

$$\int_\Omega \frac{\partial}{\partial t}\nabla\phi_m.\nabla w_j + \wp_R \int_\Omega \Delta\phi_m \Delta w_j = \int_\Omega f_3 w_j,$$

$$\int_\Omega \frac{\partial}{\partial t}\psi_m w_j - \int_\Omega \eta(\psi + \psi_{\mathrm{eq}})\Delta\psi_m w_j = \int_\Omega f_4 w_j,$$

$$\phi_m(0) = P_m(\phi(0)), \qquad \psi_m(0) = P_m(\psi(0)),$$

where P_m is the orthogonal projection in E_1 on the vector space generated by the first m vector basis $w_1, \dots, w_m$. In the equation of ϕ_m, replacing successively w_j by w_j, Δw_j, $\Delta^2 w_j$, $\Delta^3 w_j$, summing on j, using the uniform Gronwall lemma, and Lemma 5.3, we obtain, with the same techniques as in Proposition 4.1, there exists $t_3(\mathcal{B})$ such that, $\forall\, t \geq t_3(\mathcal{B})$, we have

$$|\Delta^2\phi_m(t)| \leq C, \qquad \int_t^{t+r} |\nabla(\Delta^2\phi_m)(s)|^2 ds \leq C.$$

The same techniques (from Δw_j to $\Delta^4 w_j$) applied to the equation of ψ_m yield that there exists $t_4(\mathcal{B})$ such that $\forall\, t \geq t_4(\mathcal{B})$, we have

$$|\Delta^2\psi_m(t)| \leq C, \qquad \int_t^{t+r} \left|\nabla\left(\Delta^2\psi_m(s)\right)\right|^2 ds \leq C.$$

Then, there exists C such that $\forall\, u_0 \in \mathcal{B}$, for almost all $t \geq t_3(\mathcal{B})$ and $t_4(\mathcal{B})$, we have, for $u(t) = \mathcal{S}(t)u_0$, $|\Delta^2 u(t)| \leq C$; which, with compact imbedding of E_2 in E_1, and with $u(t) \in (\mathcal{C}([0,T], E_1))^2$, implies uniform compacity of the operators $\mathcal{S}(t)$.

5.4. Existence of a Global Attractor

We are now in position to apply a theorem of [11, p. 23] and we have thus proved the following result.

THEOREM 5.4. *The semigroup $\mathcal{S}(t)$ associated to equations (1),(2) possesses a global attractor $\mathcal{A}$ in E_1.*

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